THE MATHEMATICS TEACHER

A MAGAZINE DEVOTED TO THE
INTERESTS OF TEACHERS OF MATHEMATICS

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CONTENTS

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THE MATHEMATICS TEACHER

EDITED BY
W. H. METZLER

ASSOCIATED WITH
EUGENE R. SMITH JONATHAN T. RORER

VOLUME I SEPTEMBER, 1908 NUMBER 1

FOREWORD.

The association in changing its publication from an annual to a quarterly does so with a desire to be more helpful to teachers of mathematics. A quarterly will keep the members in closer touch with the work of the association and its sections and will be of more value to them on account of the timeliness of the articles, and the additional features which can be introduced. It is hoped that many outside of the immediate territory covered by our association will find it helpful.

It will be the endeavor of those in charge to have as strong articles on the various problems confronting the teacher of mathematics as it is possible to get and such articles will be acceptable from any source.

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While the editors will hold themselves responsible for the general character of the articles which appear in the pages of this magazine, they do not hold themselves responsible for individual opinions which may appear in any article. They recognize that most questions have two sides and that the truth is arrived at by a careful consideration of a subject from all standpoints. We welcome therefore clear and concise discussions of any important topic bearing on the teaching of mathematics.

WHERE SHALL WE PLACE THE EMPHASIS?*

By W. H. Metzler.

In teaching any subject we must know what it is good for, what powers and abilities it helps to develop. We will then know at what to aim and where to place the emphasis. I am inclined to think that most teachers believe they should teach principles rather than facts, that they should aim at mental power rather than knowledge of details. At the same time many teachers seem to think they are compelled by one circumstance or another to sacrifice their ideals and best judgment. They claim that the material comes to them improperly prepared from below; that much is poor material to start with; that the demands of the colleges are heavy; that examinations covering certain ground await their students at the end of a given time; that their success is measured by the percentage of those students who pass the examinations; that one must teach in the same class, students preparing for college and students preparing for the various walks of life. All these and other causes conspire to lead too many teachers from those high ideals which they know they should and in most cases would like to follow. The justice of these claims has more or less validity under certain circumstances, but in the main I think there is no real good reason why these ideals should be departed from.

This organization should be the leader, for the territory which it covers at least, in emancipating teachers of mathematics from these yokes of bondage which draw them from proper ideals.

If the material is poor or comes to us in any stage poorly prepared we have simply to make the best of it and that is surely not done by aiming at facts but by aiming at principles and methods and to obtain as much mental power as possible.

The demands of the colleges are perhaps in some cases too heavy. Too heavy in that they lay too much stress on the facts or demand too much ground to be covered well in the allotted time. In my judgment a student who has covered two thirds or three fourths of the work well is better prepared than one who has covered the whole ground indifferently or even moder-

* Read at the annual meeting, Lancaster, Pa.

ately well. If colleges are demanding too much, which I think is not true of most of them, it would be decidedly to their advantage to demand less, and if they do not modify their demands, I think secondary schools would be justified in covering only so much of the ground as can be done well in the time. For myself I would prefer that a student come to college lacking in the quantity rather than in the quality of his mathematics, if there is to be a lack at either point.

Examinations will be ever with us, though it is to be hoped that they will not always continue to be quite so much of a bugbear as now. The bugbear will disappear as soon as students come to their examinations properly prepared. It is perhaps not usual but it is quite possible for students to enjoy examinations and they all will enjoy them when they are so prepared that they feel that they can pass any reasonable examination on the subject. Every healthy person likes once in a while to test their strength, mental as well as physical. Let those students who cannot thus be prepared well in the allotted time take longer for it. Some students have more ability than others and some are slower than others. In view of these facts, it seems strange that so many teachers expect every member of a class to do the same work almost equally well in the same time. We must not try to bring all to the same level of development. There must be development for all and the greater the ability the greater should be the accomplishment demanded, otherwise the parable of the talents has no meaning. Here I would like to put in a little plea for those students of rather exceptional ability. In a class of any size they are apt to get overlooked if indeed they are always recognized. They can without effort do their work as well or better than the rest, so they beat time and do but little. These students, few though they be in number, should have work which would exercise them and put their powers to the test; otherwise they will not develop and will never amount to much with all their ability. Does not the lower third too often set the pace for the whole class? This lower third of the class needs attention and a good deal of it, but they and the middle third should not receive it all.

If students were not allowed to come up for examination until they are prepared, then the percentage of those who pass would be high in all cases. This surely can be done if all will work for it.

Now the teaching of students preparing for all kinds of work in one class has, I think, no disadvantages connected with it. I would like to lay down the thesis that, for the most part, the best preparation for college is the best preparation for life and the best preparation for life is the best preparation for college. In making this statement I have in mind that in connection with most subjects it is the character of the work in rather than the contained facts which are of most importance. Facts have a large value, but it is training that counts for most. A few years ago it used to be said that a college education spoiled a man for business. This came from the man of business and was probably due to the fact that the college education did not equip him with the technique of that particular business. Given a chance, however, the properly trained college man soon showed that he could rapidly master the technique and that he possessed an ability and power which enabled him to forge ahead of his non-college-trained competitor, and today the man with college training is in demand for all phases of business life. This fact alone should be sufficient to prove my thesis. If we look into the case carefully we will find that that which enables the college man to succeed is not the possession of a large body of facts, but rather that ability and willingness to take any body of facts, sift them, taking those which are essential, and after careful and critical consideration produce a solution of the problem connected with them.

Among the most important things a student can get out of a course is to learn the tools to be used and the methods and principles of using those tools to solve its problems. These, together with a well-developed will to make them work and a strong character to give their work expression, are large assets in the equipment of an individual in any walk of life. All training should have for its end a broader and a larger life.

Having considered these causes one by one I trust I have shown that there exists no good reason for which teachers should forsake that which they know to be best in object and aim. To carry out our convictions in spite of difficulties we need courage and to give encouragement. The student needs much of the latter. Criticism and commendation both cultivate, the one cultivates the bad and the other the good. We should encourage by commending the good in a pupil and

thereby give them courage to do what they might otherwise be unable to do.

I should like in the time remaining to illustrate briefly from the subject of geometry something of what I mean by teaching principles. In the first place I may say that to study geometry for the mere body of facts it contains is a waste of time for most students, and I wish to add that if facts alone are aimed at little else will be acquired. I suspect that most students of geometry in the secondary schools, even if they know how to prove most of the propositions, have little real knowledge of geometrical principles and methods.

I would begin by laying down just what was necessary for a foundation and then with these tools teach the student at every point to see—(1) what tools he has, (2) which of these tools are applicable to the case in hand, (3) how to use them for its solution. Early in the course I would bring out the fact of pure logic, that to every proposition there corresponds three and but three others; its converse, its obverse and its contrapositive or, to use terms that would perhaps be better for students in geometry, its converse, its negative and its negative converse. We have thus the four propositions:

(1) If A is true then B is true,
(2) If B is true then A is true,
(3) If A is not true then B is not true,
(4) If B is not true then A is not true.

From these it may readily be seen that they go in pairs. Nos. (3) and (4) bear the same relation to each other as (1) and (2); (1) and (3) the same relation as (2) and (4); (1) and (4) the same relation as (2) and (3). It is also readily seen that the contrapositive of a proposition readily follows as a necessary consequence of the proposition. That is, it follows from pure reasoning without the use of any other propositions. Thus (4) follows from (1) and (3) follows from (2) and vice versa. It follows from this that if two, viz., the proposition and its converse, are true, then all four are true. This alone will go a long way to give system, method and direction to the work of the student. He must know just what synthesis consists of, what analysis means and how it can be used, as well as the indirect method of proof. Knowing these, he will have clear and practically complete guiding principles in the solution

of geometrical problems and not do as many students do, finish elementary geometry with the idea that it is a set of puzzles which are to be tried as a puzzle with little if any method.

These are briefly some indications of what I mean by principles and methods. A student thus equipped will have a set of tools and a knowledge of their use, so that he can carve out the solution of most any geometrical problem, whether old or new. Some of you have in your teaching been paying attention to these things and will agree with me that geometry thus studied and taught has not only great interest, but a value almost immeasurable.

Syracuse University, Syracuse, N. Y.

ELEMENTARY LOGIC AS A BASIS FOR PLANE GEOMETRY.*

By Eugene R. Smith.

This paper is the report of an experiment which I have been trying in my classes, and it will on that account necessitate a frequent use of the personal pronoun.

Some nine years ago, my attention was attracted by the numerous mistakes in elementary logic which my pupils made. I began to wonder whether logic was a natural or an acquired habit, and to be on the lookout for errors of the same kind outside my classes.

I soon found that many men in everyday life were prone to use the converse or the obverse of a known statement, if it happened to suit their purposes, without the slightest qualm. Within a short period of time I heard two striking illustrations of this. In the first place a well-known speaker demonstrated that certain conditions were always followed by certain consequences, and ended up in a burst of eloquence with the statement that since those consequences were now present in our country, the conditions must be abroad in the land.

In the second case, the debating team of one of our large high schools attempted to win a debate by the very same kind of reasoning. As one of the judges I had to disallow the entire

* Read at the annual meeting, Lancaster, Pa.

argument, and I could not but feel that in so doing, I was judging, not the boys of the debating team, but the school from which they came, for they had evidently been drilled within an inch of their lives along that very line of argument.

I then turned my attention to various mathematical texts, and I found that, while the sins of commission in this particular respect were not very numerous, the sins of omission were indeed frequent. One of the worst violators of the laws of logic gave the following brilliant example of how not to prove a theorem:

A transversal had been shown to be perpendicular to one of two lines, but not to the second. The consequent reasoning was: "If both lines had been perpendicular to this transversal they would have been parallel, but they are not both perpendicular to the transversal, so they are not parallel." There was no redeeming reference to the parallel axiom, nothing to relieve the atrocity of an assumed obverse.

I might add that my class investigations were not entirely along mathematical lines, but included such questions as this: Suppose John agrees to go to New York if Henry goes, and John is then seen in New York, is Henry necessarily there? Such questions aroused warm arguments between members of the classes, some boys seeing clearly the error of assuming the condition from the conclusion, while others failed at first to see any vital difference between a statement and the related statements.

After watching these various things for some time, I concluded that in so far as my experience showed, a correct logical sense was not the natural inheritance of the human race, but was rather the result of laborious groping after truth.

My next step was to try to improve the existing condition in my own classes, and I did this by a brief course in elementary logic at the start of plane geometry. This course was confined to the definitions of obverse, converse and contraposite in their simplest forms, and to the use of these relations in reasoning, more especially of course contraposite and converse reasoning.

After becoming accustomed to the use of the relations in simple concrete cases, the pupil was given the general case.

(1) If A, then B.

(2) If B, then A. (Converse.)

(3) If not *B*, then not *A*. (Contraposite.)

(4) If not *A*, then not *B*. (Obverse.)

The definitions of these relations are always given in as simple a form as possible, and the pupil is drilled until he can distinguish them with certainty. Such questions as " What is the converse of the observe of a statement " tests his ability to think the relations.

The next step is to establish the " law of contraposite," which is stated " The contraposite of any true statement is also true." This is done as follows:

Given the true statement " If *A*, then *B* " suppose " not *B* "; then there is the choice of " *A* " and " not *A*." But " *A* " is always followed by " *B*," which would give " If not *B*, then *A*, then *B* "; and this is a manifest contradiction. Therefore " not *A* " must be the correct conclusion, making the statement " If not *B*, then not *A*."

This proof may be required from the student or not, as a teacher wishes, but it should certainly be talked over with the class, for it is easily understood, and gives a finality to the law. This proof depends upon the fundamental law that a statement must not contradict itself, which is a special case of the law of excluded mean, but this is going too deep for use with a class.

The other important law is the law of converse, which is stated as follows:

" If statements whose conditions cover all possibilities, and no two of whose conclusions can be true at once, are known to be true, their converses are also true."

Proof.

Given the true statements

If *A*, then *X*,
If *B*, then *Y*,
If *C*, then *Z*,
If *D*, then *W*,
etc.,

where *A, B, C, D*, etc., cover all possibilities, and no two of *X, Y, Z, W*, etc., can be true at once.

Suppose *X*, then neither *Y, Z, W* nor any other conclusion can be true, for no two conclusions can be true at once. Then

by the law of contraposite no one of the conditions B, C, D, etc., can be true, and since the conditions cover all possibilities, A, the only remaining possibility must be true.

These two facts—that the contraposite of a true statement is also true, and that, while the converse of one statement can never be assumed, the converses of statements which fulfill the converse law are true—are drilled into the student by numerous examples, and are referred to in recitation often enough to keep them well in mind.

For the student who persists in assuming a converse, the following argument is usually effective:

"A donkey has a head, you have a head, therefore what must we conclude?"

The special case of the converse law, which involves greater, equal, and less, is discussed in full, and the pupil is taught that the obverse is true only when the converse is, for it is the contraposite of the converse.

It might be said that this study involves much less use of time than would be thought at first, and the time so used is more than made up before the end of plane geometry.

One of my most awful experiences occurred the first time I started teaching this to a class. I wished to use such simple logic that the text-books on the subject were of little use to me, and I was forced to make my own outline.

After having talked over the related statements with my class and having told them that the contraposite of a true statement was always true, I asked the class to make up familiar conditional statements, write their obverses, converses and contraposites, and verify our conclusions by experiment. The next day when I brought up the subject of contraposite, the brightest boy in the class raised his hand, and when called upon, told me that he had explained the lesson to his father the night before, and that his father had become so interested that they had looked up the subject in the Century Dictionary, only to find that the dictionary said that the contraposite was not always true, quoting an example to show this.

That was a heavy blow to my budding scheme, and I was forced to tell the class that I must examine what the Century said on the subject before I dared say it was wrong. I reassured them as best I might, but such an authority loomed large to all of us and the recitation was not a great success.

I hurried to the Century after school and read the passage, which was exactly as the boy had said. It did not take long, however, to see that the Century was indeed wrong, for not only was it possible to prove that the contraposite of any true statement was also true, but the example given, if looked into carefully, only confirmed this fact. The example was as follows:

"If the ink blots, I will not spill it."

Contraposite—"If I spill the ink, it will not blot."

I will spare you a discussion of the fallacy involved, but I was forced to convince that class that the Century was wrong, and when I had done it, I considered that pure logic had won a great victory. I need hardly say that a logical error in such an authority convinced me all the more that there was supreme need of some logical training.

When a class has learned to distinguish statements which are obverse, converse and contraposite, and has had impressed upon it the fact that true statements always go in contraposite pairs and that the obverse or the converse of a single statement is never safe to assume—although the obverses and converses of exclusive conditional statements covering all possibilities are true—the class is not only quite largely safeguarded against a frequent kind of error, but it is provided with a powerful weapon of attack. When I say that it is safeguarded against errors I do not take the position that mistakes of just this kind will not creep in—for they will. If there is any way to get perfect reasoning from pupils, I have not yet found it. I do mean that such errors are greatly decreased and that the ones which do occur are from carelessness, not ignorance, and are quickly recognized if the correctness of the step is questioned. A boy may add 2 and 3 and write 6—he knows better, and if the teacher questions the answer, he sees his error. So, here, if a boy assumes a wrong logical step, the question "What step in reasoning did you use?" quickly sets right the difficulty. The difference between pupils trained in this method and those wholly or even partly ignorant has been shown many times in my classes, when a pupil has entered the class after the logic had been finished. Such a pupil invariably used incorrect logic in discovering some of the new theorems, while the class as a whole distinguished clearly between the correct and incorrect

logical steps. I might explain here that my classes work out the proofs of the theorems without a text-book, so weakness in logic shows more quickly than by the other method.

As a weapon of attack, contraposite stands alone. I believe it to be, next to the general method of classification and elimination, the most important single method in geometry. Instead of wasting time in teaching it as an extra method, the teacher saves time, for its applications are almost continuous throughout the subject, and, among other things, it does away entirely with the need of "reductio ad absurdam," which is purely contraposite reasoning without the merit of being recognized as such.

I think I am safe in saying that a pupil needs just three methods of attack for plane geometry: the classification and elimination method, the analysis method—used mainly for construction work—and the method of pure logic, which includes contraposite, converse and obverse reasoning.

The contraposite, of course, requires no method—so called—for the only proof needed is to state a known theorem and say that the required statement is its contraposite and must therefore be true.

Converses and obverses of single statements have a method of their own. If one is asked to prove the converse or the obverse of a single known statement, such a proof can be possible only if the single statement is exclusive. I can best show my meaning by an example:

A line through the midpoint of one side of a triangle parallel to a second side bisects the third side.

The converse form: A line through the midpoints of two sides of a triangle is parallel to the third side.

Now this converse is true, not alone because the original statement is, but because the line obtained in the first statement is the only one through those two points; that is, there is an exclusive element which can be added to the first statement. This then is the key to the converse or obverse method when applied to a single statement:

Draw the figure according to the conditions of the original statement, then draw it according to the conditions of the required statement; finally show that the two are identical for converse, are different for obverse. This will be more clearly shown by other illustrations a little later.

The use of converse reasoning when the theorems already proved cover all possibilities is too well known to need discussion. I might say, however, that the proof usually given is not complete, for the converses can be assumed only after the statements are shown to be exclusive statements covering all possibilities, and this is usually neglected.

I will illustrate the use of these methods in proving theorems, by the use of a few of the common theorems.

The subject of parallels is an excellent example of the use of each kind of logical reasoning. The pupil can very readily see that there is an interdependence between parallels and the angles formed with a transversal, and the investigation of this relation will cover the whole subject of parallels. I always treat the angles in sets of four rather than in alternate and exterior interior pairs, thus combining all the theorems on the equality of the angles formed into four theorems. The sets of four have been shown in a preliminary theorem to depend on an alternate interior pair.

Note.—These are stated in abbreviated form.

1. If the lines are not parallel, the angles are not equal.
2. If the angles are unequal, the lines are not parallel.
   This is the converse of (1).
3. If the angles are equal, the lines are parallel.
   This is the contraposite of (1).
4. If the lines are parallel, the angles are equal.
   This is the obverse of (1).

Very early in the geometry I have taken up the theorem that an exterior angle of a triangle is greater than either interior angle not adjacent to it, and (1) is evidently this same theorem. It is naturally the first to be taken up, for it involves—not the new idea of parallels—but the already familiar idea of intersecting lines. The pupil sees at once that when two lines are crossed by a transversal, a triangle is formed if the lines meet, and the inequality of the two alternate interior angles follows.

But the other three statements are all related to this one, and (3) is true at once because it is contraposite to (1).

There remain to prove only (2) and (4). These can be proved separately by the converse and obverse methods, or one can be proved, and the other will follow because they are contraposites. I usually prove (2) as the obverse of (3) and use

contraposite reasoning to obtain (4). If this is done, the entire subject of parallels is covered by the exterior angle theorem, which is of value for other purposes, and one obverse proof, the other statements being contraposites.

The obverse method applied to (2) would be as follows:

If the angles are equal, the lines are parallel (3).

If a line is drawn through the same point on the transversal, making the angles unequal, it cannot be parallel, for there can be but one parallel to a line through a point.

To prove (4) by the converse method, the following proof could be used:

Given two parallels crossed by a transversal; if a line is drawn through the point where one of the parallels crosses the transversal, making equal angles with the transversal, that line will be parallel to the other of the two given parallels (3), and therefore will coincide with the given line through the same point, by the parallel axiom.

(2) and (4) are excellent examples of the method to be applied in proving the converse and the obverse of a single statement, and these proofs are probably both well worth talking over with a class, even though the use of contraposite makes but one of them necessary.

Two very simple examples of contraposites depend upon axioms.

"Two straight lines can intersect in but one point" is the contraposite of "Through two points there can be but one straight line."

"Lines parallel to the same line are parallel to each other" is proved by the contraposite of "Through a point there can be but one parallel to a given line."

As good an example of the use of the law of converse as occurs in the geometry is in the discussion of the interdependence of the sides and the angles of a triangle. Suppose that the two theorems, "If two sides of a triangle are equal, the opposite angles are equal," and "If two sides of a triangle are unequal, the opposite angles are unequal, the greater angle being opposite the longer side," have been proved; these facts can be arranged in this form, using a general triangle ABC:

If $AB > BC$, $\angle C > \angle A$;
If $AB = BC$, $\angle C = \angle A$;
If $AB < BC$, $\angle C < \angle A$.

These statements evidently cover all possibilities ($>$, $=$, $<$), and no two of the conclusions can be true at once; therefore their converses are true, and

$$\text{If } \angle C > \angle A, AB > BC;$$
$$\text{If } \angle C = \angle A, AB = BC;$$
$$\text{If } \angle C < \angle A, AB < BC;$$

which proves the two theorems in which the condition refers to the angles, the conclusion to the sides.

Whatever kind of reasoning a pupil is using he should first state definitely the known statements from which he is reasoning, and then should draw his conclusions, stating what kind of reasoning is being used. In using the law of converse, he should say that the conditions cover all possibilities, and that no two of the conclusions can be true at once before drawing the conclusions.

In conclusion I may say that the experiment has been a great success. I not only find the reasoning of my classes more logical—in fact more critically logical—but the classes discover new facts with much greater facility than before.

The records of pupils who have mastered these first principles of logic have been excellent, both in college entrance examinations, and in more advanced work taken up by many of them, and I feel that I have greatly strengthened my teaching of geometry by my experiment in adding elementary logic to the course taught.

POLYTECHNIC PREPARATORY SCHOOL,
BROOKLYN, N. Y.

MATHEMATICAL HISTORICAL MATERIAL FROM THE EAST.*

BY BERTHA LILLIAN BROOMELL.

It is impossible to give in a few words any adequate idea of the rare and valuable mathematical material collected last year in the Far East by Professor David Eugene Smith while on his sabbatical leave. We can merely mention some of his most important acquisitions.

* Abstract of paper read at the annual meeting, Lancaster, Pa.

Among these are about four hundred volumes of printed mathematical works and two hundred manuscripts from Japan, including the great mathematical classics of that country; a set of *sangi*, the sticks which were used to represent the coefficients in the solution of numerical equations; and numerous native instruments for surveying and other purposes.

From China were brought between two and three hundred volumes, including the great encyclopedia of mathematics of about 1650 and the famous logarithmic table of 1712 printed at Peking. There is in manuscript the translation of Euclid made about 1600 by Matthew Ricci, numerous interesting dials, and a beautiful modern manuscript on silk of an ancient work showing the use of the magic square.

There are over one hundred Sanskrit manuscripts from India, the most famous being the *Lilávati* of Bhaskara, one copy of which dates from about 1400. Of numerous mathematico-astronomical works, one rare one of over one thousand pages was written before 1500. There is an interesting block book printed at Llassa from Thibet, and twenty or thirty palm-leaf manuscripts from Burmah, South India, and Ceylon, some thirty Persian and Arabian manuscripts of great value, including a translation of Euclid written in 1300, and other works on arithmetic, algebra, geometry and trigonometry, besides numerous photographs showing the great gnomons and other features of the Hindu astronomers, and the early numerals in the inscriptions of northern India. A set of a dozen plaster casts show the magic number squares cut in stone in Chittagong, east of Calcutta.

To his collection of dice, a subject Professor Smith has carefully studied in connection with the history of the number concept, he has added sixty-five different specimens, ranging from 600 B. C. to 1700 A. D.

A large number of European works on mathematics were also secured.

TEACHERS COLLEGE,
NEW YORK CITY.

INTERNATIONAL COMMISSION ON THE TEACHING OF MATHEMATICS—PRELIMINARY REPORT IN OUTLINE.*

By Bertha Lillian Broomell.

In the unavoidable absence of Prof. David Eugene Smith the following report was given:—

The section, "Philosophy, History and Education," of the Fourth International Congress of Mathematicians held at Rome, April 6–11, 1908, made to the Congress on the initiative of Professor David Eugene Smith, of New York, a proposition to create an international commission for the purpose of making "a comparative examination of the methods and plans of the study and teaching of mathematics in the secondary schools of the different nations."

The congress adopted the resolution and appointed Professor Felix Klein, of Göttingen, Professor Sir Alfred George Greenhill, of London, and Professor H. Fehr, of Geneva, to organize such a commission and to report to the next congress in Cambridge, England, in 1912.

These gentlemen have since adopted a general plan of organization and work, including the representation on the commission of the different countries, the provision of necessary funds, the official organ of publication and the official languages of correspondence and reports, the general aim of the commission, and the method and scope of its work.

The commission will consider the present conditions of organization and method of mathematical instruction, the aims of such instruction and the branches taught in the various kinds of schools, the methods of conducting examinations in the different countries, and the preparation of teachers. It will also consider the modern tendencies in mathematical teaching in regard to the inter-relation of subjects, the connection between mathematical and other studies, the practical applications of mathematics, the history of the subject, etc. It will also investigate the question of positive dangers in current reform movements in order to guard against these.

Teachers College,
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* Read at the annual meeting, Lancaster, Pa.

CHECKS, THEIR USE AND ABUSE.*

By William E. Breckenridge.

In this paper I shall endeavor to answer the following questions:

1. Is it worth while to check mathematical work systematically?

2. What are some dangers to be avoided in introducing the checking system and making it mandatory upon both teachers and pupils?

The scope of the discussion is restricted to secondary schools with some slight reference to industrial and trade schools. The facts presented are gathered from a four years' experience in the practical use of checks in the classrooms of one of the large high schools of New York City.

Notwithstanding the vigorous paragraphs in the books of Professors Smith and Young urging the use of checks, I know only one school where every pupil is held responsible for checking his work and where the teachers are expected as a department to enforce this responsibility on the pupil. Spasmodic attempts here and there have come to my attention where a teacher would talk rather enthusiastically about checking certain particular kinds of work, but the idea of systematically attending to this matter has not yet been adopted.

It is a common complaint from those who employ high school or college men in mathematical work that it is very rare to find a man trained to habits of accuracy.

The ruling of the Regents of New York State on marking inaccuracies on examinations may be interesting. Dr. Wheelock, Chief of the Examination Division, in an address delivered before the New York Section of the Association last year, stated that it was a rule of his department not to give over 70 per cent. for an answer that was incorrect. The College Entrance Board has no rule, but leaves the marking to the judgment of the individual examiner. The Civil Service Board of New York City deducts somewhat less for a mechanical error than the Regents.

* Read at the annual meeting, Lancaster, Pa.

The attitude of secondary teachers, as far as the author's observation has extended, seems to be rather more lenient in marking inaccuracies than the Regents.

Text-books are improving somewhat in the treatment of checks. Several recent books have ventured to extend the checking system to subjects unchecked in previous books. But there is no book that I have seen that persistently and systematically urges the checking of all work, even in those subjects that admit of checks aside from the answer book.

And yet there is great need of more accurate work in our schools. A habit of accuracy is essential to success in applied mathematics.

No surveyor would venture to proceed with his calculations until he had checked them carefully.

An actuary must not only be sure of his work, but must know what methods of interpolation are least liable to error under given conditions.

In banking, an error of addition may cause the loss of a thousand dollars, if made in a certain column.

The industrial chemist must be able to weigh accurately and check his results. Precise observation as well as accurate computation are demanded here.

I fancy few teachers would deduct much, if anything, on account of an error made in copying down an example, yet such an error in industrial work is as fatal to success as an error in theory. Business men complain that boys are not able to copy a set of numbers accurately.

Of course there is no such thing as absolute accuracy. Even after a problem has been checked there always exists the possibility of the corresponding error, but very rarely does this occur and a habit of accuracy will diminish the probability of its occurrence.

I believe that one of the greatest needs of our mathematical teaching at present is a higher standard of accuracy on the part of teachers and pupils which may be attained largely through a general, systematic and persistent use of checks.

A subject like this needs to be worked out in the classroom. All teachers admit the use of checks in a general way, but you will find upon asking whether they teach them, that they have no plan for their systematic use, that their pupils are not

held strictly responsible for checking work, and that, on the whole, the regular work at present takes the whole time, so that the check is considered as something which is perhaps desirable, but under present conditions rather impractical.

The common remark is "It takes too much time." Of course it takes time and patience to develop habits of accuracy, but is it not better to reduce the quantity of work, if necessary, in order to improve the quality of the essentials?

The tendency is to reduce the quantity of work required and improve the quality of what is left. The omission of infinite series from the list of subjects in advanced algebra by the College Entrance Examination Board, on the advice of the American Mathematical Society, and the entire omission of advanced algebra from the entrance requirements to Columbia are examples of this tendency.

My theme, then, is that a pupil should be taught to check all work until the habit is so firmly fixed that the pupil will not be happy nor satisfied with a piece of work till it is checked.

Most of the theories in regard to checks presented in this paper are familiar to you, but it is their effects on the pupils as observed during four years of experiments in the classroom that furnish my material for new arguments for the general adoption of the checking system.

Mathematical work should be checked for two reasons, first, the effect on the pupil and second, the effect on the quality of work done. A boy who has developed a habit of reviewing and examining all work with respect to its accuracy has a self-confidence that is difficult to develop in any other way.

On the other hand, it is very depressing to a student to be in constant doubt as to the accuracy of his work and very injurious to repeat his errors, day after day, without suspecting that the errors exist. He becomes confirmed in bad habits. He may make errors, but he should know that he has made them and not think his work right when it is wrong.

Another reason for checking is that it does secure greater accuracy. Of course checks are not the only means of securing accuracy; skillful teaching and drill are essentials. But it is the general opinion of the teachers who have tried systematic checking that the time taken is well spent.

Another effect on the quality of the work is that incidentally

many important points of theory are taught which are not likely to receive as much emphasis unless the work is checked.

The distinction between the general values of the letters in a polynomial and the unique value of the unknown in a linear equation is usually unknown to a pupil who does not check his work and find out that general values will not satisfy the equation. Extraneous roots are not so well understood as they would be if every root were checked. The convention of taking only the principal root of a radical is not generally understood even by writers of text-books. At least two of the more recent books bring this out clearly. All the others would be compelled to do so, if they should check their results. This is especially important in the problems in quadratic form involving radicals.

If graphs are checked by algebraic solutions, the one-to-one correspondence between algebra and geometry may be illustrated.

The present ambiguous use of zero both as the elementary symbol and as an infinitesimal is illustrated in checking division by avoiding the use of such particular values as will render the division equal to zero.

In trigonometry checks will illustrate the approximate nature of results from logarithmic tables, the calculation of errors, and their proper distribution.

Should all work be checked? Yes; understanding that sometimes work done under the teacher's immediate supervision to emphasize particular points of the subject is sufficiently checked by the teacher's supervision, provided always that the pupil knows how to check it.

The home work should certainly be checked, also all tests and class work not under the immediate supervision of the teacher and often even that to make sure that the student knows how to handle the check.

Continuity and persistency throughout all kinds of work are essential to the development of the proper attitude of the pupil toward checking.

Professor Bain, in his chapter on habit, says: "Continuity of training is the great means of making the nervous system act infallibly right." This applies to checking as well as to drill.

What are the characteristics of a good check?

It should be independent of authority outside of the student himself, such a check as would make the presence of the teacher or answer book unnecessary. Without this quality, the student has no method of testing his results after he leaves school.

Again, the best check tests the problem by some other process than repeating the work already done. The mind tends to act again through the former pathway of discharge and is likely to repeat the error if the same route is taken that was travelled at first.

A chemist in a leading industrial chemical company says that it is a principle of quantitative analysis that chemical work repeated within twenty-four hours is of very little value. In careful experiments, at least a day is allowed to elapse before repeating the work. This same chemist uses an independent method of checking his weights. He counts the value of the holes from which he took his weights and then counts his weights in the scale pan. One checks the other so far as the correct observation and adding of the weights is concerned.

Hence subtraction is better checked by adding subtrahend and remainder or by substituting numerical values than by going over the work of subtraction again.

In trigonometry the solution of a triangle when two sides and the included angle are given is best checked by a sine formula not previously used in the problem. If the third side c has been found by using A, use the sine formula involving B and C for the check.

The ideal check should be readily applied. When the check becomes tedious and difficult of application, much of its value is lost. The substitution of larger numerical values than unity for the letters in checking multiplication is usually unnecessary since nearly all errors are made in the coefficients.

When several checks are possible one should be selected which is most readily applied and which is most effective in detecting errors. Pupils may be allowed to use their judgment and determine by competitive tests in the class which check is most quickly applied and which is most likely to detect errors.

The particular problem should be studied with respect to its special difficulties and the check used which will detect errors that are likely to occur. For example, the substitution of numerical values in the case of factoring involving a common

monomial factor is not so good as multiplying since if the letters are each taken equal to unity, the exponents are not checked and errors are likely to occur in exponents in this case. The multiplication check is more rapid than the substitution of larger values than unity for each letter.

What are the different kinds of checks? First there is the answer book. I know at least one school of high mathematical scholarship where the answer book is put into the hands of pupils in elementary algebra and continued throughout the course.

Another check is the partial answer. For example, a problem in square root has the result given to two figures when three or four are required.

Other forms of checks are the substitution of numerical values, symmetry, homogeneity, approximate estimates, repeating the work, and, for arithmetical computations, the slide rule or a more complicated counting machine.

What are the best checks for the several kinds of work in algebra and trigonometry?

For the four fundamental operations with polynomials, the substitution of particular values is the best check even in subtraction where it is possible to add the remainder and subtrahend, and in division where the quotient may be multiplied by the divisor. The reason is, first, this check is more quickly applied. A trial by the watch will convince of this if the letters are each given the value of unity. This is usually sufficient since errors are not likely to occur in exponents.

Second, this check is more likely to be done by the student than the other, especially on home work. It is something which can be written down by the pupils and quickly inspected by the teacher.

The other check should be explained, but this one should be used in nearly all cases.

Some examples in division should be given which have a remainder in order to give the student a full understanding of the relation between dividend, divisor, quotient and remainder. The ordinary text-book in algebra gives only examples where the division is exact. From constant use of such examples the pupils come to think that the test of accurate work is that there is no remainder—as they say, " It comes out even."

In factoring, the monominal factor case should be checked by multiplication, since here errors are likely to occur in the exponents. Other cases, however, lend themselves very readily to the numerical substitution check.

Of course care must be taken to avoid those particular values which make a factor zero.

In the removal of complicated systems of parentheses two methods of work may be taught, one as a check on the other. Whether the method of removal beginning at the outermost parentheses is favored by the teacher, one may be used as a check on the other, or, better yet, a third method of removing all the signs at once may be taught and used as check on either of the other two methods. As far as I know, this last method, while not new, is not given in any text-book; so I venture to illustrate it as follows:

$$a + \{\, 2a - \overset{1}{[}3a - \overset{2}{(}4a - \overset{3}{\overline{5a - 6}})]\}$$

Neglect the positive signs of aggregation. Number the other signs according to the number of minus signs which affect the terms immediately within the sign of aggregation. Remove all signs of aggregation in one step, changing the signs of the terms immediately within the odd-numbered signs of aggregation and leaving unchanged the signs of those immediately within those with even numbers.

$$\begin{aligned} &a + 2a - 3a + 4a - 5a + 6 \\ &= -a + 6 \end{aligned}$$

A more complicated example involving two distinct systems of parentheses follows:

$$\begin{aligned} &a + \{\, 3a - \overset{1}{[}2a - \overset{2}{(}6a - 1) + 4a - \\ &\qquad\qquad \overset{2}{(}7a + 1)]\} - \overset{1}{[}3a - \overset{2}{(}4a - 1)] \\ &= a + 3a - 2a + 6a - 1 - 4a + 7a + 1 - 3a + 4a - 1 \\ &= 12a - 1 \end{aligned}$$

A similar method holds when the parentheses have coefficients.

This system is as readily mastered as either of the others. The truth about this kind of work is, however, that while it is

still required by examining boards, in my opinion it has not enough value, to justify it as a topic in the syllabus. The syllabus in algebra of the Regents of New York State reads: "Examples free from ingenious repetition of complications should be selected for practice."

A study of recent papers set by the Regents or by the College Entrance Board shows a tendency to omit tests on complicated work of this kind.

In numerical linear equations, of course, substitution of the value of the unknown is the only check. The distinction between this unique value of X and the general values of A, B, C, etc., in a polynominal should be emphasized at this point by means of the check. Some students in the class will check by substituting unity for X. Let them find their error and teach them the reason for the distinction.

Checks for literal equations may be either the substitution of numerical values for the known letters or the substitution of the root itself. When the substitution of the root is tedious, the particular numerical value should be used. The check for problems involving the making of equations is the analysis of the original problem with the results secured. Substitution in the equation of course merely checks the solution of the equation and not its correct formation.

In H.C.F. and L.C.M., care should be taken to avoid the substitution of numerical values, since such substitution does not check the work. The H.C.F. of $a + b$ and $a - b$ is 1, but if $a = 4$ and $b = 2$, the H.C.F. is 2. In this work the only check good for general use is to repeat the work. However, if only two numbers are involved and both H.C.F. and L.C.M. are required, a useful test of accuracy is the formula H.C.F. $\times$ L.C.M. = the product of the two numbers.

All through the reduction of fractions, additions, subtraction, multiplication, division and complex fractions, the check of numerical values is effective. Here particularly text-books fail in sufficiently emphasizing the importance of testing the difficult work of handling fractions. More students fail right here in their algebra than in any other part of the subject. Careful drill and checking of work are very necessary.

When graphs are introduced along with simultaneous equations, the geometric solution should be checked by the algebraic

result. The one-to-one correspondence of geometry and algebra may be illustrated by the use of the check at this point.

The numerical check applies also to involution and evolution. In evolution again, as in division, examples should be given that do not "come out even," as the students say.

For the theory of exponents, the answer book or repeating the work are the proper checks. The answer book is useful also in the transformation of radicals. Radical equations need to be checked carefully to detect extraneous roots. This subject of extraneous roots and that of equivalent equations are too little treated in most text-books.

If the authors of most of our text-books in elementary algebra should check the equations in quadratic form involving the radicals, with the answers from their own answer books, and should use the convention regarding principal roots, a great many of their results would be found to be wrong, since they would not check.

In the substitution case of simultaneous quadratics it is a sufficient test of the work to substitute in one equation only, taking care not to use the one from which the second unknown was secured. Any error in the problem that is likely to occur will appear by this process.

In the homogeneous case, it is sufficient to check the first and third sets of answers, since the checks for the other sets will be a repetition of this work. This work is difficult and somewhat tedious at times, but it is well worth while both for the sake of accuracy in the problem itself and for the review of radicals and imaginaries that it affords.

For the progressions where the work is not too tedious, the series should be written out. The value of the repeating decimal may, of course, be checked by the converse process of reducing the fraction to a decimal.

Partial fractions should be checked by addition. Complex numbers may be illustrated by graphs and checked algebraically.

Determinants applied to equations may be tested by the algebraic solution.

For Horner's Method there are the checks of substitution obtaining approximately zero for the function, or the theory of the relation of the roots and coefficients. This theory applies also to quadratics.

Incommensurable roots obtained by graphs may be tested by substitution.

In trigonometry all text-books check the solution of the triangle when three sides are given, but very few indicate what methods should be used in other cases.

My experience with trigonometry is that it is well worth while to check all triangles right and oblique. The check for a right triangle is the formula $b^2 = c^2 - a^2$ (c being the hypotenuse) in the fractured form $b^2 = (c + a)\ (c - a)$.

For the oblique triangle, given two sides and the included angle, the sine formula should be used which involves the third side of the triangle and is a formula not previously used in the work; *e. g.*, if we have given the angle C and sides a and b, we may find the side c by using the sine formula involving the letters c and a. If so, we must check by the sine formula involving the letters c and b. When three sides are given, of course the check is the sum of the angles $= 180°$.

In the other cases of oblique triangles the tangent formula is the only reliable check formula. This is somewhat tedious in application, but even so, it is worth while to spend the necessary time upon it, since it furnishes a more reliable test of accuracy than going over the work a second time.

The difficulty with the check by repeating the operation is that the mind tends to act again in the same way it has once acted. Hence an error once made may not be readily found. In the case of the solution of the triangle when three sides are given, the amount of error should be clearly stated on the student's paper and the distribution among three angles clearly indicated. Right here a good transit is a great help in clarifying the subject of possible approximation to accuracy. When a boy has used a transit to some extent he gains an increased respect for accuracy both in seeing and in computing. He learns that a result that is inaccurate is worthless whether it comes from careless reading of the instrument or from careless computation.

Trigonometric equations should be checked by substitution in the original equation. Here again when radicals enter, extraneous roots creep in. Hence the check is a necessary part of the work.

The accuracy of the regular formulas may be tested by the substitution of a particular value for the angle such as 30°, 45°

or 60° whose functions are well known. Functions of the double angle may be tested by assuming $x = 30°$, while for the half angle x should be taken $= 60°$.

So much for checks of algebra and trigonometry in secondary schools.

Just a word in regard to the use of the check in continuation schools.

The kind of men who work in our evening schools, especially those who are in the industrial or the trade schools, are particularly appreciative of methods that make them more efficient workmen. Nearly all of them are employed during the day and feel keenly any advantage of knowledge which makes them better workmen or which enables them to secure employment at an advanced salary. I have been very intimately associated with this class of young men for the past three years. With them I have used all the checks which I have described in algebra and trigonometry and in addition work with the slide rule. If checking were too tedious and not worth the time spent these men would very quickly have shown it, but the fact is they have been very proud of their newly acquired ability to test their own work. These men judge a thing entirely from the standpoint of its use to them. If you have an experiment in teaching and wish to judge it from the standpoint of interest, no better place can be found in which to test it than the classes of men in evening high and trade schools.

But there are certain things to be guarded against in the use of checks. Their abuse may lead to disaster in the scholarship of a class.

First, avoid making the check tedious. Rather review the direct operation or use the answer book than cumber the subject too much with needlessly long operations which take much longer than the problem itself. Just when a check becomes tedious is a fine point for the teacher's judgment to decide.

Second, the check is only one element in obtaining accuracy. Teaching and drill must not be neglected. A student must know how to obtain his results before he will have any results to check. I have observed teachers using so much time during the recitation for checks that there was not enough taken for teaching and drill on the processes. This, however, is no argument against the use of checks, but rather a poor adjustment of the plan for teaching the lesson.

A third difficulty in using checks is the spasmodic nature of some of the experiments. A teacher becomes partially inspired with the idea of checking work and without thinking out the whole matter carefully enough, he proceeds to check work now and then. He teaches his class how to check, but does not immediately hold them responsible for doing so. At first a class must be held up to checking in the same way they would be held to any other kind of work. It involves extra labor and time spent on the home work, which few boys will do at first unless required by the teacher. After the student is more mature, and has seen the value of checking, he will demand methods of testing his work and will not need to be held up to it.

This latter stage, however, does not come to any great extent until a boy is actually employed in real work outside the school.

Again some pupils will hand in daily work that looks as though it were checked, but on test work the results are wrong. A search for the reason for this reveals the fact that the student has "made a bluff at the check," as he says. He has not really tested his results on his daily work, but put down something that looks like a check to deceive the teacher. For this reason it is necessary to observe carefully the result of the test work and if wrong, to mark such dishonest work very severely. When a boy sees that he cannot deceive by his dishonesty, he wil usually abandon his attempt.

For the prevention of these abuses I have found certain precautions effective.

First, a certain per cent. should be deducted for failure to check work. This should be decided upon by the department and as rigidly used as possible. In my own school this is 20 per cent. That is, we deduct 20 per cent. from all work that does not show the check.

In a large department there will be some classes at the introduction of such a standard that will not be able to attain it at once. This will be true because of the poor teaching in that particular class. In such cases some elasticity should be allowed in marking, but such elasticity should be noted by the head of the department and the teacher brought up to the standard as soon as possible.

I do not contend for a fixed 20 per cent. Perhaps this is too little, perhaps too much. One teacher in my school holds his

classes up to 25 per cent. But I do contend that some fixed per cent. should be adopted and the teacher as well as the pupils held up for it.

Another device for securing the use of checks is to have a uniform method of ruling paper, so as to reserve a column for checks. If a good-sized column at the right of the paper is ruled and reserved for checks only, the student will be apt to fill out the blank space with the checks.

Too long lessons must be avoided. If checking is required, fewer problems must be assigned both on home work and for tests. It is better to have a few well-selected examples done and checked than to have more done without the student's knowing whether they are right or wrong.

The value of a thing in the long run is determined by common sense. Common sense is a hard thing to define. Perhaps as good a definition as any is one given at a meeting of our New York Section last year by a member of the Board of Examiners of New York City. He said: "Common sense is that which appeals to the non-professional."

Judged by this standard checks seem to be very favorably received.

In conclusion I wish to emphasize the importance of teachers of mathematics insisting upon checks in all mathematical calculations for the sake of (1) the good habits and the feeling of self-confidence in the pupil, (2) knowledge of certain points of theory not gained so well in other ways, (3) increased accuracy in work.

The great thing to be aimed at is to develop in the student such habits of invariably reviewing his work and considering his result in the light of the check that best suits it, that he will not be happy nor content until he has done this whenever a new problem presents itself. If this is done, systematically and invariably, there will be developed a degree of accuracy and a feeling of self-confidence that cannot be obtained in any other way. It is a result which is carried over into other affairs of life.

I hope to see an organized effort on the part of our association to request those who set the examinations for the C.E.E. Board, the Regents of New York State, and the colleges to require checks of all problems on the examinations which they

set and to deduct a fixed percentage for failure to do this. They should insist that the candidate should show on his paper that he understands how to check his work. Of course this will not be done without adequate notice to the schools.

When it is done we shall have text-books adequate for such a requirement, teachers will systematically teach checks, and the result will be a greatly increased accuracy and efficiency in our mathematical work.

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THE AIMS IN TEACHING GEOMETRY AND HOW TO ATTAIN THEM.*

BY W. E. BOND.

The aims in teaching geometry should be, according to my views:

1. That the pupil should acquire an accurate, thorough knowledge of geometrical truths.

2. That he should develop the power of original, logical, geometrical reasoning.

3. That he should acquire a habit of thought which will give him a practical sagacity; which will develop his judgment, increase his resourcefulness, and fit him to cope more successfully with the many and varied problems of his after life; which will teach him to take a many-sided view of things, so that if the avenue of attack is blocked, he shall be able to promptly, cheerfully and successfully attack from another quarter.

In my opinion the last-mentioned aim is by far the most important; the first, as an end in itself, I deem the least important of the three.

Having stated the aims to be attained, I now turn to a discussion of the difficulties which, in my own experience, have presented themselves, and have tended to hinder the attainment of these aims.

In my early experience as a teacher of geometry the first difficulty I met was the seeming lost or dazed condition of so

* Read at the meeting of the Syracuse Section, December 29, 1908.

many of my pupils for the first few weeks of the term. They did not see what we were trying to do. Their previous work in mathematics had been so different from this that it hindered rather than helped them. Not understanding what they were trying to accomplish, they naturally did not make much headway in accomplishing it. With some the mists cleared away in a few days; with others in a few weeks; while still others remained in the densest kind of a fog for the whole term, or dropped out, discouraged, by the way.

The second difficulty I encountered was the *memorizing* of demonstrations. In spite of changing the lettering of the figures, changing the shape of the figures, turning them upside down, etc., a large number, especially among the girls, would depend more upon *memory* than upon *reason*. Even where the demonstration as a whole seemed to be logically worked out, a little questioning would often bring out the fact that certain positions were accepted upon the authority of the author, and had not been clearly thought out by the student.

A third difficulty was in teaching the pupil how to attack new propositions. No set rules or methods or suggestions seemed to help them much. What they needed was a *condition of mind,* which could not be taught, directly.

These were the main difficulties which at first hindered the realization of my ideals, and they have been ever since, and are today, the problems which I am trying to solve. I shall now proceed to indicate along what lines I am trying to work out these problems.

In the first place I have become convinced that the first difficulty I mentioned, the dazed condition of the pupils at first, is due to a lack of understanding of what *proof* or *demonstration* consists; of what it really means; and of the necessary process to be gone through in proving a proposition. So, first of all, I believe we should get our classes to see that proof consists in showing that the proposition in question is a special or particular case of some general truth which must have been already admitted without proof. Whether it takes one day, or three days, or a week, the teacher of geometry should not begin the regular work of the subject until every pupil is perfectly clear on this point. So that when we ask them if they will admit, without proof, certain statements which we call "axioms,"

every member of the class will know just *why* we ask them to admit, without proof, these truths, and just what relation these axioms are to bear to the work which is to follow.

As to the third difficulty, that of teaching pupils how to attack new propositons, I have come to the conclusion that they can learn to do only by *doing*. The first step in learning to swim is to get into the water. The teacher of Latin does not put a translation of Cæsar into the hands of his pupils; why should we start off by putting a pony into the hands of ours? We have all noticed that actual grappling with originals seems to give the greatest increase in power to cope with originals. Then why not let the work be original reasoning from the start? If in this way we can overcome the third difficulty I mentioned, the second, the memorizing tendency, will take care of itself.

So, several years ago I began experimenting along this line. I rearranged the theorems of the first book of plane geometry so that, while in a logical order, they were somewhat different from that of any text-book. I cut out most of the corollaries, the few really important ones being incorporated as theorems. I tried to arrange the theorems so that the proofs on which a theorem depends should precede it as closely as possible. Also I tried as far as practicable to grade the theorems from easy to difficult. I did not attempt to group the theorems according to any classification of the figures; for example those which prove triangles equal are numbers 4, 5, 9, 19, 31 and 32 in my list. My first experiment was with a class of about forty. The result, compared with the results of my previous teaching, was such that I have since continued it.

Each pupil is supplied at the start with a note-book, but at first with no text-book. After explaining to the class the nature of the subject-matter in geometry, or of what geometry treats, the teacher states to the class that he wishes to prove to them certain propositions about lines, angles, etc., and asks them to tell him what method of procedure he must follow. If the preliminary work has been properly done, the class will assure him that he must first get them to admit, without proof, some simple, self-evident, general truths,, and then must show that the propositions he wishes to prove are special, particular cases of these general truths which they have already admitted. Cautioning them to consider carefully and not admit anything

unless it is so simple and self-evident that they cannot help admitting it, the teacher proposes, one at a time, the list of axioms which he has prepared. As fast as admitted the pupils are told to write them in their note-books; labeling them axiom 1, axiom 2, etc.

The pupils now know why these axioms have been proposed, and are anticipating the theorems which are to follow. They do not feel lost, or uncertain as to what it all means, but are ready and waiting for the course of reasoning that is to come.

The teacher now proposes theorem I, and proceeds to demonstrate or prove it to them. When all have signified their acceptance of the proof, the teacher dictates the demonstration, the pupils write it in their note-books, and are required to be able to reproduce it next day, *verbatim*. In this way the first five or six theorems are dictated by the teacher, written down and reproduced by the class. The object of this dictation and verbatim reproduction is to furnish the pupil a model so that in his future work he may follow a logical, well arranged form. The next four or five theorems are demonstrated by the teacher, but not dictated. By this time the class should be ready to begin to walk alone, the teacher gradually removing the leading strings, being careful not to discourage them by requiring more than they can do at first. In the more difficult propositions the teacher may for a time suggest the construction lines, but this prop should also be removed as soon as possible. For the first eight or ten weeks most of the advance lesson is prepared in class under the eye of the teacher. Fifteen or twenty minutes before the close of the period the advance theorem is assigned, the pupils take pencil and paper and attack the proposition, always first of all writing down what is *given* and what they are to *prove*. Let us suppose the theorem to be "Two straight lines which are parallel to the same straight line are parallel to each other," which is no. 16 in my list. The teacher passes up and down the aisles and observes the progress each is making. Those who are evidently getting a start he leaves with perhaps a word of encouragement, and halts at a desk whose occupant has as yet not written a word or drawn a construction line and asks what is the trouble. The pupil replies that he can't "get started." The teacher asks what ways he has had of proving lines parallel. The pupil names Definition

of parallel lines and Theorems 7, 10, 12 and 14. The teacher observes that from such a number of ways of proving lines parallel he certainly ought to find one that can be made to apply to this case, and suggests that he try them one after the other until one is found that will work. The teacher stops again at the desk of a pupil who is started along a wrong line and asks a question which calls the pupil's attention to his error. In this way the teacher goes around the room, giving encouragement here, a hint there, but no help except where imperatively needed. At the close of the period the papers are collected, errors marked in blue pencil and returned to the pupils next day. (For this work I use a blank form somewhat similar to the *Geopad*. The average pupil will in from three to five minutes fill in one of these blanks a review demonstration which would require fifteen minutes to write out in full.)

At the beginning of the next recitation the teacher puts the figure for this theorem on the board, the different ways of demonstrating it are discussed, criticisms are invited, and questions answered. In this way the pupils form habits of study which will fit them to later prepare their lessons at home.

Upon the board, and also copied in each pupil's note-book, there should be a blank form, or table of about a dozen columns, with the following headings: Angles Equal, Angles Unequal, Lines Equal, Lines Unequal, Lines Parallel, Lines Perpendicular, Triangles Equal, etc. Each heading has a double column, the first marked "Direct Proofs," the second "Indirect Proofs." As each theorem is demonstrated it is recorded in the appropriate column; also definitions and axioms which might be used in proving the truth of the headings. The pupils soon learn to consult this list for the available proofs to be used in demonstrating new propositions, and as all the previous proofs are recorded in their appropriate columns, the pupil is almost sure if he perseveres, to find one that will apply.

By the time the first book has been finished in this way, the pupils have acquired good habits of thought and have learned how to study. They may now, if preferred, be supplied with text-books, or they may continue as in Book I. The pupils themselves, if given a voice in the matter, will invariably choose to go on without a text-book. If a text is used the demonstra-

tions must be slightly modified to fit Book I as the pupils have had it. This they easily do, and it serves as a further check to memorizing demonstrations.

What are the disadvantages in this method of teaching geometry without a text-book? I have found only two. First, unless the teacher is watchful, pupils may fall into the habit of using mathematical terms in a loose and inaccurate way; second, pupils who have never used a text-book have more difficulty in following a text-book demonstration than those who have used one. For both these reasons I think it is advisable for the *inexperienced* teacher to use a text for the last part of the course.

Teachers who do not feel competent or do not have the time to rearrange the theorems of Book I may take them from some other text-book to which the students do not have access. All texts cover practically the same ground in the first book.

The matter of enthusiasm and interest I have not yet mentioned, yet they are by no means the least benefits to be derived by this method. Pupils will become as much absorbed in succeeding with a demonstration as in winning a game. Time and again I have heard a student say at the successful ending of a struggle with a theorem that he had "beaten the old cat." There is always a wholesome rivalry to see who will first succeed in getting a demonstration. And the one who has to confess to a failure always feels it a keen disgrace.

This method encourages and develops the individuality of the pupil. It makes him an independent thinker. It gives him confidence in his own powers.

The teacher, also, is more enthusiastic, and becomes more interested in his work. He becomes better acquainted with his pupils and their individual peculiarities of mind. He learns what their difficulties really are, and hence can aid them more intelligently.

I have tried this method on more than twenty classes, ranging in numbers from twenty to sixty. I have also during this time started several different assistants in this method, usually inexperienced teachers fresh from college. From this experience I am satisfied that any teacher who can get fair results with a text-book, can get much better results with original work.

To summarize:

Aims.

First. Acquirement of knowledge of geometrical truths.

Second. Power of logical geometrical reasoning.

Third. Develoment of sagacity.

Difficulties.

First. Dazed condition in early part of the work.

Second. *Memorizing* instead of *reasoning.*

Third. Inability to attack new propositions.

Remedies.

For the first difficulty. Preliminary discussion of the nature of proof.

For the second and third. Original work.

POTSDAM NORMAL SCHOOL,
POTSDAM, N. Y.

WHAT SHOULD BE THE AIMS IN TEACHING ALGEBRA AND HOW ATTAIN THEM.*

BY ARTHUR M. CURTIS.

The aims in teaching algebra are largely those that govern the teaching of any subject:

The first and most essential aim must be to enlist the interest of the pupil in the subject to the end that he will put forth *persistent* effort in its mastery.

From the nature of the subject there are three special aims that may well claim the attention of the teacher:

1. To increase the pupil's knowledge of mathematical shorthand.

2. To increase the pupil's knowledge of truth, mathematical truth, in preparing him for future work in college and technical schools.

3. To establish in the mind of the pupil the firm belief that he can *of himself* distinguish truth from error and to see to it that he *forms the habit of proving all things.*

The means are the following:

* Abstract of a paper read at the meeting of the Syracuse Section, December 29, 1908.

1. The teacher, thoroughly believing in the aims and strong enough to execute for them.

2. Pupils possessing a reasonable degree of aptitude.

3. A school wherein the spirit of earnest effort predominates.

4. A book—almost any book with plenty of problems *and* a *teacher.*

Oneonta Normal School,
Oneonta, N. Y.

NOTES AND NEWS.

The Mathematics Teacher is sent free to every member of the association who has paid his dues, and it is hoped that each member may interest another in the magazine and the association and send their names to the secretary for membership.

The Annual Meeting was held at Franklin and Marshall College, Lancaster, Pa., November 28, 1908.

The treasurer's report, showing a balance of $358.53 on hand, was audited and found correct.

The chairman of the Algebra Syllabus Committee reported the work well under way, and asked for a time extension of one year in which to finish the syllabus. The request was granted, but the sense of the meeting was that the work should be in finished shape by the next annual meeting.

The following delegates and committees were appointed:

Delegates to the American Federation—L. S. Hulbert, Daniel D. Feldman, I. J. Schwatt, Eugene R. Smith, J. T. Rorer, David Eugene Smith, Edwin S. Crawley, W. H. Metzler, W. A. Cornish, William Betz.

Committee on Publication—William H. Metzler, chairman and editor-in-chief, Jonathan T. Rorer, Eugene Randolph Smith.

Committee to Investigate the Present Condition of Mathematics in Continuation Schools—William E. Breckenridge, chairman, two others to be appointed.

Proposed Amendment.—Paragraph I of Section VI to read: "The annual meeting shall be held at a time and place to be selected by the Council."

Officers Elected for the Year 1909.

President—William Henry Maltbie, Woman's College, Baltimore, Md.

Vice-president—William E. Breckenridge, Stuyvesant High School, New York City.

Secretary—Eugene Randolph Smith, Polytechnic Preparatory School, Brooklyn, N. Y.

Treasurer—Emma Hazleton Carroll, High School for Girls, Philadelphia, Pa.

Council Members—William H. Metzler, Susan C. Lodge.

The next meeting will be held at Syracuse University, Syracuse, N. Y., on Saturday, April 10.

The following new members were elected:

R. BENNETT, WILLIAM M., A.M.; West High School, Rochester, N. Y. 350 Bronson Ave.

S. BOND, WILLIS ELMER, A.B.; Potsdam Normal School, Potsdam, N. Y.

P. BOWMAN, INA C., A.B.; High School for Girls, Philadelphia, Pa 2029 North 17th Street.

P. CONSTABLE, MARY LOUISE, B.S.; Girls High School, Philadelphia. Pa. 1241 S. Broad Street.

N. Y. DANIELS, HARRIET MCDOUAL, A.B.; Charlton School, New York City. Union Settlement, 237 E. 104th Street.

N. Y. DAVIS, MARTHA GROSVENOR, B.S.; Brooklyn Heights Seminary Brooklyn, N. Y. 414 W. 118th Street, New York City.

P. EVANS, PROFESSOR HENRY BROWN, Ph.D.; University of Pennsylvania, Philadelphia, Pa. College Hall.

S. LIDELL, BURTON WILLIAM, A.B.; Potsdam Normal School, Potsdam, N. Y.

N. Y. MAIER, G. W. MARQUE, M.S.; Polytechnic Preparatory School. Brooklyn, N. Y.

Pt. MALLENAUER, ADELINE, A.B.; Perryopolis High School, Perryopolis, Pa.

P. PATTON, MABLE CAREY; Girls High School, Philadelphia, Pa Media, Pa.

PECK, PROFESSOR NOBLE, A.M.; The George Washington University, Washington, D. C. 1718 22d Street, N. W.

R. PIERCE, EUNICE MARTHA, A.B.; Lockport High School, Lockport, N. Y. The Genesee.

N. Y. SEARS, JONATHAN THATCHER, A.B.; Polytechnic Preparatory School, Brooklyn, N. Y.

THE SYRACUSE SECTION held its third annuel meeting on the afternoon of December 29, 1908.

The following program was given before a very interested body of teachers:

I. "Field Notes on the Teaching of Secondary Mathematics," by Inspector H. De W. De Groat, Education Department, Albany, N. Y.

II. "What Should be the Aims in Teaching Algebra and How to Attain Them," by Arthur M. Curtis, Oneonta Normal School, and Charles Earle Biklé, Syracuse High School.

III. "What Should be the Aims in Teaching Geometry and How to Attain Them," by Willis E. Bond, Potsdam Normal School, and Eugene P. Sisson, Colgate Academy.

There were about one hundred in attendance. Fourteen applications for membership were received.

Officers for the ensuing year were elected as follows:

President—Mr. Eugene P. Sisson, Colgate Academy.

Secretary and Treasurer—Daniel Pratt, Syracuse University.

Executive Committee—Mr. Fred R. Keck, Utica High School; Mr. Wm. V. Wilmot, Huntington High School; Miss Anna I. Byrne, Syracuse High School.

THE NEW YORK SECTION held a regular meeting at the High School of Commerce, New York City, Friday, December 11, 1908, at 8:15 P. M. The following program proved very interesting: Report of Secretary and Treasurer; Report of the Committee on Marking Mathematical Papers, A. Latham Baker, Manual Training High School; General Topic for the Evening: Modern Tendencies in the Teaching of Algebra, or, An Evening with the Writers of some of the Modern Text-books in Algebra. (*a*) Fletcher Durell, Mathematical Master in the Lawrenceville School; (*b*) Isaac J. Schwatt, Assistant Professor of Mathematics in the University of Pennsylvania; (*c*) N. J. Lennes, Instructor in Mathematics, Brown University; the discussion was led by the following: (*a*) Wm. E. Breckenridge, Stuyvesant High School, New York City, (*b*) Merle L. Bishop, Boys' High School, Brooklyn, (*c*) Oscar W. Anthony, DeWitt Clinton High School, New York City; election of two members of the Executive Committee.

THE COUNCIL OF THE AMERICAN FEDERATION OF TEACHERS OF THE MATHEMATICAL AND NATURAL SCIENCES met in Baltimore, Md., on Monday, December 28. Of the thirty-three members nineteen were present either in person or by proxy. The report of the Executive Committee, presented by Mr. J. T. Rorer, outlined the work of the year in connection

with organization, the appointment of a special committee on the bibliography of science teaching, the issue of the November BULLETIN and the preparations for the Council meeting at Baltimore. It pointed out some of the specific questions which might naturally engage the attention of the officers of the Federation during the coming year.

The following article was added to the articles of federation as adopted at the Chicago meeting:

These articles may be amended at any annual meeting of the Council by a two thirds vote of the members present provided notice of the proposed amendment has been sent to all members of the Council and to the president and secretary of each federated society at least thirty days prior to the meeting.

The following resolutions were unanimously passed:

Resolved: That the Americal Federation of Teachers of the Mathematical and Natural Sciences respectfully urges upon the Congress of the United States the enactment of such legislation as will greatly increase the scope and importance of the United States Bureau of Education and will enable it to render immediate and effective aid in the promotion of education in the mathematical and the natural sciences; and

Resolved: That the Executive Committee of the Federation be authorized to take such steps as may seem to it desirable to further such action by Congress.

Three committees of the Federation were authorized as follows: One on a syllabus of propositions in geometry; one on publication and publicity to report next year on the present needs and facilities for publishing material of interest to the federation, and to make such recommendations as to ways and means of improving these facilities; the third, a committee to investigate the present conditions of college entrance, to define the attitude of the Federation toward college entrance problems, and to recommend action that may tend to unify and simplify the college entrance requirements.

The following officers for the coming year were elected: *President,* H. W. Tyler, Massachusetts Institute of Technology; *Secretary-Treasurer,* C. R. Mann, University of Chicago; *Other Members of the Executive Committee,* G. W. Hunter, De Witt Clinton High School, New York; J. T. Rorer, Central High School, Philadelphia, Pa.; C. H. Smith, Hyde Park High School, Chicago.

Pd.

Pd.